Logarithmic superdiffusivity of the 2-dimensional anisotropic KPZ equation

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We study an anisotropic variant of the two-dimensional Kardar-Parisi-Zhang equation, that is relevant to describe growth of vicinal surfaces and has Gaussian, logarithmically rough, stationary states. While the folklore belief (based on one-loop Renormalization Group) is that the equation has the same scaling behaviour as the (linear) Edwards-Wilkinson equation, we prove that, on the contrary, the non-linearity induces the emergence of a logarithmic super-diffusivity. In fact, our result is that the correlation length $\ell(t)$ grows like $\ell(t) \sim \sqrt{t} \times (\log t)^{\delta/2}$ for t large, for some $\delta \in (0, 1/2]$ (in contrast with the Edwards-Wilkinson case where $\delta = 0$) and we argue that actually $\delta = 1/2$. Even if unexpected in the present context, this phenomenon is similar in flavour to the super-diffusivity for two-dimensional fluids and driven particle systems.

I. INTRODUCTION

Stochastic growth phenomena are ubiquitous in non-equilibrium statistical physics [1]. Over the last 20 years most of the attention has focused on one-dimensional ($1d$) growing interfaces (e.g. the boundary of a bacterial colony spreading in a two-dimensional medium). Experimental, theoretical and mathematical results succeeded in unveiling the universal features (most notably, scaling exponents and non-Gaussian limiting distributions) of what is by now known as the $1d$ KPZ universality class. Also in dimension $d \geq 3$ progress was made in both the physics and mathematics literature and recently the prediction [2] of asymptotically Gaussian behaviour for small coupling constant has been rigorously established. Instead, the harder case of $2d$ growth, on which we focus here, is still to a large extent unexplored. We study an anisotropic version of the $2d$ KPZ equation for which we determine super-diffusive behaviour, contradicting the claim of diffusivity repeatedly made in the previous literature.

The KPZ equation is the stochastic partial differential equation

$$\partial_t H = \frac{1}{2}\Delta H + \lambda \langle \nabla H, Q \nabla H \rangle + \xi\,, \tag{1}$$

where the real-valued random height function $H = H(t,x)$ depends on time $t \geq 0$ and on a d-dimensional space coordinate x, Δ is the $d-$dimensional Laplacian, ξ is the space-time Gaussian white noise, i.e. $\mathbb{E}\xi(x,t) = 0$, $\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta(x-y)\delta(t-s)$ ($\mathbb{E}(\cdots)$ denoting the average), Q is a fixed $d \times d$ symmetric matrix and the coupling constant $\lambda \geq 0$ tunes the strength of the non-linearity (here, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$).

The equation was introduced in a seminal paper by Kardar, Parisi and Zhang [2], that focused on the situation in which Q is the identity matrix, thus reducing the non-linearity to $|\nabla H|^2$. In this case (1) is also connected to the partition function Z of a $(d+1)$-dimensional directed polymer in a random potential (the time variable is the $(d+1)$-th space coordinate) via the transformation $Z = \exp(\lambda H)$. For general Q, (1) serves as a model for $(d+1)$-dimensional stochastic growth, the non-linear term encoding the slope-dependence of the growth mechanism, and it is presumed to arise as the scaling limit of a large class of interacting particle systems. The phenomenological connection with microscopic growth models is the following: for (1) to correctly describe the height fluctuation process around a macroscopically flat state of slope $\rho \in \mathbb{R}^d$, one should take $Q = D^2 v(\rho)$, where $v(\rho)$ is the average speed of vertical growth and $D^2 v$ is the $d \times d$ Hessian matrix of v.

A natural problem associated to (1) is to determine whether the non-linearity is relevant or not in a Renormalization Group (RG) sense, i.e. whether large-scale features of the equation, such as roughness and growth exponents α, β respectively defined as

$$\begin{aligned} &\mathrm{Var}(H(t,x) - H(t,y)) \sim |x-y|^{2\alpha} \\ &\mathrm{Var}(H(t,0) - H(0,0)) \sim t^{2\beta}\,, \end{aligned} \tag{2}$$

for t and $|x-y|$ large, differ or coincide with those of the solution H_{EW} of the linear Edwards-Wilkinson (EW) equation corresponding to (1) with $\lambda = 0$. It has been argued in [2] and confirmed since then in many works [3–7] that the non-linearity is relevant in dimension $d = 1$ (the growth exponent changes from $\beta_{EW,d=1} = 1/4$ to $\beta_{KPZ,d=1} = 1/3$), whereas it is not if $d \geq 3$, provided λ is smaller than a critical threshold $\lambda_c(d)$ (the mathematical proofs of this [8–12] require that $Q = \mathbb{I}$). Notice that these results can be at least guessed on the basis of a formal scaling argument, which goes as follows. Let $\varepsilon > 0$ and rescale H diffusively, i.e. set

$$H^\varepsilon(t,x) := \varepsilon^{1-\frac{d}{2}} H(t/\varepsilon^2, x/\varepsilon)\,. \tag{3}$$

Then, H^ε solves

$$\partial_t H^\varepsilon = \frac{1}{2}\Delta H^\varepsilon + \lambda \varepsilon^{\frac{d-2}{2}} \langle \nabla H^\varepsilon, Q \nabla H^\varepsilon \rangle + \tilde{\xi} \tag{4}$$

where $\tilde{\xi}$ is again a space-time white noise equal in law to ξ in (1). From (4), it is immediate to see that, as ε goes to 0, for $d = 1$ the coefficient of the non-linearity blows up, thus suggesting relevance, while for $d \geq 3$ it goes to 0. In 2 dimensions, however, the situation is more subtle: the equation is scale-invariant so that a priori the non-linearity is dimensionally marginal and the qualitative behaviour of (1) was predicted in [1, 13] to depend on the sign of the determinant of Q. In the case of $\det Q > 0$ the non-linearity changes the growth and roughness exponents (see e.g. [14]) to two universal values $\alpha_{KPZ,d=2} \approx 0.39...$, $\beta_{KPZ,d=2} \approx 0.24...$, compatible with the exact scaling relation $\alpha + z = 2$, with $z = \alpha/\beta$ the dynamic critical exponent. Instead, for $\det Q \leq 0$, which is called "anisotropic KPZ" (AKPZ) and includes both the linear equation $Q = 0$ as well as models of growth of vicinal surfaces [13], the exponents should be the same as for the EW equation, i.e. $\alpha_{EW,d=2} = \beta_{EW,d=2} = 0$, with logarithmic instead of power-like fluctuation growth in (2). This has been conjectured on the basis of one-loop RG computations [1, 13] and supported by numerical simulations [15] of a discretized version of (1). Further, it has been claimed [1, 13, 15] that the large-scale fixed point of (1) is the EW equation. The purpose of the present work is to disprove the latter claim. Indeed our main result is that if $d = 2$ and $Q = \mathrm{diag}(+1, -1)$ is the diagonal matrix with entries $(+1, -1)$ then, as soon as $\lambda \neq 0$, (1) is *logarithmically super-diffusive*, namely the correlation length $\ell(t)$ behaves like $\sqrt{t} \times (\log t)^{\delta/2}$ as time grows, for some $\delta > 0$, while EW has the usual diffusive growth $\ell(t) \sim \sqrt{t}$. [The correlation length is defined by the condition that the correlation function $S(t, x)$ in (7) is small for $|x| \gg \ell(t)$.] Interestingly, the exponent δ does not continuously go to zero as $\lambda \to 0$ and, in fact, a mode-coupling theory computation (which is presented below) suggests that $\delta = 1/2$ for every $\lambda \neq 0$. A more precise statement of the results, together with an idea of the proof, is given below. A full mathematical proof can be found in [16]. Before we proceed, let us remark that, even though in the context of $2d$ growth our findings were unexpected, logarithmic corrections to the diffusive scaling have already been observed (and rigorously proved) for other two-dimensional out of equilibrium systems such as driven particle systems (see [17, 18] for the asymmetric simple exclusion process on $\mathbb{Z}^2$, in which case though the value of δ is 2/3), fluid models (see [19]) and self-repelling polymers (see [20, 21]) where $\delta = 1/2$.

II. THE AKPZ EQUATION: MAIN RESULTS

A distinguishing feature [22] of the $2d$ equation (1) with $Q = \mathrm{diag}(+1, -1)$, i.e. the AKPZ equation

$$\partial_t H = \frac{1}{2}\Delta H + \lambda[(\partial_{x_1} H)^2 - (\partial_{x_2} H)^2] + \xi \qquad (5)$$

is that it has a Gaussian log-correlated stationary state η. More precisely, η is a zero-mean Gaussian field (GFF in the mathematical jargon) whose covariance is $\mathbb{E}(\eta(x)\eta(y)) \sim -\log|x - y|$ (with $x = (x_1, x_2)$), showing a vanishing roughness exponent. Note that the stationary state is independent of λ.

The equation (5) is mathematically ill-posed: the solution at fixed time is a GFF, that is merely a distribution, so that the square $(\partial_{x_i} H)^2$ does not make sense. A usual way out (that was already adopted implicitly in [2]) is to regularize the equation. In [16], we replaced $(\partial_{x_i} H)^2$ by $\Pi((\Pi\partial_{x_i} H)^2$, where Π is a cut-off in Fourier space, that removes all modes $|k| \geq 1$. The non-linearity then becomes $\mathcal{N}(H) = \Pi((\Pi\partial_{x_1} H)^2 - (\Pi\partial_{x_2} H)^2)$. As observed in [23], the stationary state of the regularized equation is still the GFF η and from now on we work with the stationary process with initial condition $H(0) = \eta$. We expect that our results would hold unchanged if we regularized the noise instead, as is often done. Also, in [16] we work on a torus of side length $2\pi N$ instead of the infinite plane, and N is sent to infinity before any other limit is taken. For lightness, we drop the N-dependence in the formulas below.

A convenient way of encoding the growth in time of the correlation length is through the *bulk diffusion coefficient* $D_{bulk}(t)$ [24]. For the KPZ equation, the usual way to define it is as in [5]. In our context, we let $U(t, x) = (-\Delta)^{1/2} H(t, x)$ (this operation just means that, in Fourier space, each Fourier mode $\hat{U}(t, k)$ is given by $|k|\hat{H}(t, k)$), that solves a $2d$ stochastic Burgers equation whose stationary state is simply the Gaussian white noise ρ, with $\mathbb{E}\rho(x) = 0, \mathbb{E}(\rho(x)\rho(y)) = \delta(x - y)$. Then, D_{bulk} reads

$$D_{bulk}(t) = \frac{1}{2t}\int_{\mathbb{R}^2} |x|^2 S(t, x)\, dx\,, \qquad (6)$$

with

$$\begin{aligned} S(t, x) &= \mathbb{E}[U(t, x)U(0, 0)] \\ &= \mathbb{E}[U(t, x)U(0, 0)] - \mathbb{E}[U(t, x)]\mathbb{E}[U(0, 0)] \end{aligned} \qquad (7)$$

where the last equality holds because at fixed time $U(t)$ is a centered random field with $\mathbb{E}U(t, x) = 0$. Note that we are looking at H and U at stationarity so that $S(0, x) = \delta(x)$, and $t \times D_{bulk}(t)$ measures the spread of correlations in time in a mean-square sense. The EW equation is linear, Gaussian and has all Fourier modes evolving independently and so does U_{EW} defined as above. Therefore it is possible to compute S_{EW} explicitly and show that $D^{EW}_{bulk}(t) = 1$ independently of t, corresponding to the usual $\sqrt{t}$ growth of correlation length. Our main result is that in contrast, as soon as $\lambda \neq 0$, there exists $0 < \delta \leq 1/2$ such that

$$(\log t)^{\delta} \leq D_{bulk}(t) \leq (\log t)^{1-\delta} \qquad (8)$$

for t large (to be precise, (8) is proven in the sense of Laplace transforms, see (16) below). While we do not pin down the precise value of δ, we can prove that δ *does not tend to zero as* $\lambda \to 0$, while as mentioned it equals zero for $\lambda = 0$.

Another natural question for stochastic growth processes is how they behave under rescaling. The $2d$ EW stationary equation is well known to be scale-invariant under diffusive scaling, i.e. for every $\varepsilon > 0$, H^ε_{EW} in (3) equals H_{EW} in law. Our second result shows that, the solution H of the non-linear equation (5) with $\lambda \neq 0$ is different from H_{EW} not only for $\varepsilon > 0$ fixed, but also asymptotically for large scales as $\varepsilon \to 0$. Namely, we prove that the fields $H^\varepsilon(t,\cdot)$ and $H^\varepsilon(0,\cdot)$ already decorrelate at times of order $|\log\varepsilon|^{-\delta} \ll 1$ for $0 < \delta \le 1/2$ as above. We quantify this by verifying ([16, Th. 1.2]) that given a smooth test function φ and letting H^ε_φ be the centered random variable $H^\varepsilon_\varphi(t) = \int_{\mathbb{R}^2} \mathrm{d}x \varphi(x) H^\varepsilon(t,x)$, the normalized covariance

$$\frac{\mathrm{Cov}(H^\varepsilon_\varphi(t), H^\varepsilon_\varphi(0))}{\mathrm{Var}(H^\varepsilon_\varphi(0))} = \frac{\mathbb{E}[H^\varepsilon_\varphi(t) H^\varepsilon_\varphi(0)]}{\mathbb{E}[H^\varepsilon_\varphi(0)^2]} \tag{9}$$

is strictly smaller than 1 for $t \approx |\log\varepsilon|^{-\delta} \ll 1$, *uniformly as* $\varepsilon \to 0$. This result again indicates that the large scale behaviour of (5) differs from that of EW.

We emphasize that the above *does not contradict* the numerical findings of [15], but only its conclusion that the solution of (5) shows a "very rapid, unrelenting and nearly immediate crossover to the EW fixed point". In fact, [15] numerically observes $\sqrt{\log t}$ instead of a power-like (see (2)) growth of height fluctuations in time, which is the same as for EW. This is in agreement with rigorous results, see Theorem 1.5 in [16], but it does not address the question of logarithmic corrections to the diffusive scaling or to D_{bulk}, which turns out to be the feature that really distinguishes between the EW and AKPZ classes.

III. MODE-COUPLING APPROXIMATION

Before explaining how we prove (8), let us briefly give a heuristics, based on a mode-coupling approximation [17, 25], which moreover leads to the conjecture $\delta = 1/2$. Let $\hat{S}(t,k) = (2\pi)^{-2}\mathbb{E}[\hat{U}(t,k)\hat{U}(0,-k)]$, $k = (k_1,k_2)$ be the Fourier transform of S. A direct computation shows that $\hat{S}$ satisfies the exact identity (see [16, App. B] for details)

$$(\partial_t + \frac{|k|^2}{2})\hat{S}(t,k) = -\frac{|k|^2\lambda^2}{(2\pi)^6}\int_0^t \mathrm{d}s\, e^{-\frac{|k|^2}{2}(t-s)} \int \mathrm{d}p \int \mathrm{d}q K_{p,k-p} K_{q,-k-q} \mathbb{E}\left[\hat{U}(s,p)\hat{U}(s,k-p)\hat{U}(0,q)\hat{U}(0,-k-q)\right] \tag{10}$$

where $K_{p,q} = (p_2q_2 - p_1q_1)/(|p|\,|q|)\mathbb{I}_{|p|,|q|,|p+q|\le 1}$ comes from the Fourier representation of the non-linearity in (5) and the indicator function is due to the Fourier regularisation induced by the cut-off Π. To get an (approximate) closed equation for $\hat{S}$, we perform a Gaussian approximation which allows to replace the average in the second summand at the right-hand-side of (10) by a Gaussian one. In other words, one assumes $\hat{U}$ to be Gaussian and, consequently, is allowed to replace $\mathbb{E}[\hat{U}_1\hat{U}_2\hat{U}_3\hat{U}_4]$ (with $\hat{U}_1 = \hat{U}(s,p)$ etc.) with the sum of three terms, corresponding to the possible contractions of four indices, of the form $\mathbb{E}[\hat{U}_i\hat{U}_j]\mathbb{E}[\hat{U}_k\hat{U}_l]$. The conservation of momentum then readily implies that the terms $\mathbb{E}[\hat{U}(s,p)\hat{U}(s,k-p)]$, $\mathbb{E}[\hat{U}(0,q)\hat{U}(0,-k-q)]$ are non-zero only if $k = 0$ and therefore they do not contribute, because of the pre-factor $|k|^2$ in (10). Altogether, one obtains

$$(\partial_t + \frac{|k|^2}{2})\hat{S}(t,k) = -\frac{2|k|^2\lambda^2}{(2\pi)^4}\int_0^t \mathrm{d}s\, e^{-\frac{|k|^2}{2}(t-s)} \times \int \mathrm{d}p (K_{p,k-p})^2 \hat{S}(s,p)\hat{S}(s,k-p)\,. \tag{11}$$

We now make the Ansatz

$$\hat{S}(t,k) = \hat{S}(0,0) e^{-\frac{|k|^2}{2}t - c|k|^2 t(\log t)^\delta}\,, \tag{12}$$

for k small and t large. Notice that in this regime $k-p \approx p$, which means $(K_{k-p,p})^2 \approx 1$, and $\exp(-|k|^2(t-s)/2) \approx 1$. Hence, computing the left and right hand side of (11) with $\hat{S}$ as in (12) and then equating them, we are led to

$$-|k|^2(\log t)^\delta \approx -|k|^2\lambda^2(\log t)^{1-\delta}$$

which imposes the choice $\delta = 1/2$.

IV. LOGARITHMIC DIVERGENCE OF D_{bulk}

The actual proof of (8) given in [16] starts by rewriting the bulk diffusion coefficient in its Green-Kubo formulation

$$D_{bulk}(t) = 1 + \frac{2\lambda^2}{t}\mathbb{E}\Big[\Big(\int_0^t \mathrm{d}s \overline{\mathcal{N}}(U(s))\Big)^2\Big] \tag{13}$$

where $\overline{\mathcal{N}}(U(s))$ is the spatial average of $\mathcal{N}(H(s,\cdot)) = \mathcal{N}((-\Delta)^{-1/2}U(s,\cdot))$, $\mathcal{N}(H(s,\cdot))$ being the non-linearity of AKPZ. More explicitly, $\overline{\mathcal{N}}(U(s))$ is given by

$$\overline{\mathcal{N}}(U(s)) = \int \mathrm{d}p K_{p,-p}\hat{U}(s,p)\hat{U}(s,-p)\,. \tag{14}$$

Now, thanks to [23, Lemma 5.1] and the fact that U is a stationary Markov process whose law at every fixed time is that of the spatial white noise ρ, the Laplace transform in t of $t \times D_{bulk}(t)$, which we denote by $\mathcal{D}_{bulk}$, can be written as

$$\begin{aligned}\mathcal{D}_{bulk}(\mu) &= \int_0^\infty dt e^{-\mu t} t D_{bulk}(t) \\ &= \frac{1}{\mu^2} + \frac{1}{\mu^2}\mathbb{E}[\overline{\mathcal{N}}(\rho)(\mu-\mathcal{L})^{-1}\overline{\mathcal{N}}(\rho)]\end{aligned} \tag{15}$$

with $\mathcal{L}$ the generator of U and where the expectation is taken with respect to the law of the stationary state ρ. In the Laplace transform sense, the relation (8) above for large t is equivalent to

$$\frac{1}{\mu^2}|\log\mu|^\delta \le \mathcal{D}_{bulk}(\mu) \le \frac{1}{\mu^2}|\log\mu|^{1-\delta} \tag{16}$$

for μ small.

To have a better understanding of the expectation in (15), recall that the bosonic Fock space associated to ρ can be decomposed as $\oplus_{n\ge 0}\Gamma L^2_n$, $\Gamma L^2_n = L^2_{sym}(\mathbb{R}^{2n})$ being the space of square integrable functions which are symmetric in their n two-dimensional coordinates, endowed with the usual L^2-scalar product $\langle\cdot,\cdot\rangle_n$ (see [3] for the definitions in a context similar to ours). Physically, ΓL^2_n can be viewed as the set of states of n indistinguishable bosonic particles. Let us remark that, by (14) $\overline{\mathcal{N}}(\rho)$ is clearly quadratic in ρ and it has mean 0 in view of the symmetry of the non-linearity of (8). The Fock representation of $\mathfrak{n}$ in momentum space reads

$$\hat{\mathfrak{n}}(p,q) = \delta_{p+q=0}K(p,q) \tag{17}$$

so that $\mathfrak{n} \in \Gamma L^2_2$. Let P_n be the orthogonal projection onto $\Gamma L^2_{\le n} = \oplus_{j\le n}\Gamma L^2_j$ and set $\mathcal{L}_n = P_n\mathcal{L}P_n$. It turns out (see [16, Lemma 3.1]) that the sequence $b_j(\mu) = \langle\mathfrak{n},(\mu-\mathcal{L}_j)^{-1}\mathfrak{n}\rangle_2$, satisfies

$$b_3(\mu) \le b_5(\mu) \le \cdots \le b_4(\mu) \le b_2(\mu) \tag{18}$$

and

$$\lim_{j\to\infty} b_j(\mu) = b(\mu) := \langle\mathfrak{n},(\mu-\mathcal{L})^{-1}\mathfrak{n}\rangle_2\,,$$

where the right hand side equals the expectation in (15). Therefore, in order to prove (8), it suffices to determine suitable upper and lower bounds for $b_{2j}(\mu)$ and $b_{2j+1}(\mu)$, respectively. To do so, note first that the symmetric part of $\mathcal{L}$, $\mathcal{L}_0$, acts in Fock space as $-\frac{1}{2}\Delta$ so that, for any n, $\mathcal{L}_0$ leaves ΓL^2_n invariant (i.e. conserves the total number of particles) and in momentum space it corresponds to the usual multiplication by $(|k_1|^2+\cdots+|k_n|^2)/2$. On the other hand, the antisymmetric part $\mathcal{A}$ can be written as the sum of $\mathcal{A}_+$ and $\mathcal{A}_-$, which are such that $\mathcal{A}_+^\dagger = -\mathcal{A}_-$ and the former maps ΓL^2_n to ΓL^2_{n+1} while the latter to ΓL^2_{n-1} (i.e. $\mathcal{A}_+$ and $\mathcal{A}_-$ respectively increase and decrease the number of particles by 1). If we recursively define the operators $\mathcal{H}_j$'s as

$$\begin{aligned}\mathcal{H}_3 &= -\mathcal{A}_-(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_+ \\ \mathcal{H}_j &= -\mathcal{A}_-(\mu-\mathcal{L}_0+\mathcal{H}_{j-1})^{-1}\mathcal{A}_+\end{aligned} \tag{19}$$

we obtain the alternative representation

$$b_j(\mu) = \langle\mathfrak{n},(\mu-\mathcal{L}_0+\mathcal{H}_j)^{-1}\mathfrak{n}\rangle_2\,. \tag{20}$$

From (19) it is immediate to verify that, for all j, $\mathcal{H}_j$ leaves each of the ΓL^2_n's invariant. In order to treat the inverse of $\mu-\mathcal{L}_0+\mathcal{H}_{j-1}$ and get meaningful estimates for b_j, we need to control the $\mathcal{H}_j$'s in terms of explicit multiplication operators which act diagonally in momentum space, as $\mathcal{L}_0$ does. To give a glimpse of how the procedure works, let us consider $\mathcal{H}_3$. Testing it against a n-particle state $\phi\in\Gamma L^2_n$ and using the explicit expression for $\mathcal{A}_\pm$ in [16, Eq.'s (2.18)-(2.19)], we get

$$\begin{aligned}\langle\phi,\mathcal{H}_3\phi\rangle_n \sim& \lambda^2\int \mathrm{d}k_{1:n}|k_{1:n}|^2|\hat{\phi}(k_{1:n})|^2 \\ &\times\int\mathrm{d}p\frac{(K_{p,k_1-p})^2}{\mu+\frac{|p|^2+|k_1-p|^2+|k_{2:n}|^2}{2}}\end{aligned} \tag{21}$$

with $k_{1:n} = (k_1,\ldots,k_n)$ and $|k_{1:n}|^2 = |k_1|^2+\cdots+|k_n|^2$. Using $(K_{p,k_1-p})^2\le 1$ and performing the integral on p, one obtains an upper bound of the form

$$\lambda^2\int\mathrm{d}k_{1:n}|k_{1:n}|^2|\hat{\phi}(k_{1:n})|^2\log\Big(1+(\mu+\tfrac{|k_{1:n}|^2}{2})^{-1}\Big)$$

which implies

$$\mathcal{H}_3 \lesssim (-\mathcal{L}_0)\log(1+(\mu-\mathcal{L}_0)^{-1})\,, \tag{22}$$

where the inequality above has to be understood in the sense of operators. Thanks to the structure of (19), we can iteratively bound the $\mathcal{H}_j$'s starting from that on $\mathcal{H}_3$ in (22) and ultimately attain

$$\mathcal{H}_{2j+1} \lesssim C^{2j+1}(-\mathcal{L}_0)\frac{\log(1+(\mu-\mathcal{L}_0)^{-1})}{T_{j-1}(\mu-\mathcal{L}_0)} \tag{23}$$

$$\mathcal{H}_{2j+2} \gtrsim \frac{1}{C^{2j+2}}(-\mathcal{L}_0)T_j(\mu-\mathcal{L}_0) \tag{24}$$

(see [16, Theorem 3.3] for the precise statement), where $C>1$ is a constant due to the approximations made at each step of the iteration but uniformly bounded (from above) for λ small, and the function T_j is defined as a Taylor expansion truncated at level j, i.e.

$$T_j(x) = \sum_{\ell=0}^{j}\frac{(\frac{1}{2}\log\log(1+x^{-1}))^\ell}{\ell!}\,. \tag{25}$$

Plugging (24) and (23) into (20), choosing μ sufficiently small and j sufficiently large depending on μ and C ($j\approx C^{-2}\log\log(1/\mu)$), (16) follows with $\delta\approx 1/C^2$.

Let us remark that already (22) implies a divergence of the bulk diffusivity, of order at least $\log\log t$ for t large. Indeed, plugging it into (20) with $j=3$ and exploiting the explicit expression for $\mathfrak{n}$ in (17), one sees that

$$b_3(\mu) \gtrsim \int\mathrm{d}p\frac{(K_{p,-p})^2}{\mu+\frac{|p|^2}{2}(1+\log(1+(\mu+\frac{|p|^2}{2})^{-1}))}\,. \tag{26}$$

Carefully evaluating the integral yields $b_3(\mu)\gtrsim\log\log(1/\mu)$, from which the claim follows at once.

V. GENERALIZATIONS

The methodology exploited above, which allows to establish logarithmic super-diffusivity, is robust and its applicability is not restricted to Eq. (5). We first point out that, as remarked in [22], any equation related to (5) by a rotation of the plane has again Gaussian steady states; therefore, we can immediately conclude logarithmic superdiffusivity for each of these examples which in particular include

$$\partial_t H = \frac{1}{2}\Delta H + \lambda_1[(\partial_{x_1}H)^2 - (\partial_{x_2}H)^2] + \lambda_2\partial_{x_1}H\partial_{x_2}H + \xi$$

where $\arctan(\lambda_2/\lambda_1)$ is the rotation angle. More generally, these techniques allow to analyse systems of two-dimensional coupled equations with Gaussian steady states, as those in [22, Sec. VI], which describe crystalline or magnetic growth.

One of the strengths of our method is that it goes beyond growth phenomena and we expect it to apply to other two-dimensional superdiffusive out of equilibrium systems. As mentioned in the introduction, a diffusivity of order $t \times \sqrt{\log t}$ is expected for self-repelling Brownian polymers [20, 21], for particle diffusion in random divergence-free vector fields [26] and for two-dimensional fluids [19]. In all these three cases the rigorous results available at the moment show merely a lower bound of order $t \times \log\log t$ [26, 27] which corresponds to the very first step of our iteration. A generalization of our operator bounds (23)-(24) should allow to upgrade this estimate from log log to a logarithmic scale.

VI. CONCLUSIONS

We have studied the AKPZ equation, an anisotropic variant of the $2d$ KPZ equation (1) with $\det Q < 0$, which has a Gaussian, logarithmically rough stationary state. The common folklore belief is that it has the same scaling behaviour as the (linear) EW equation. While indeed our results confirm that both have the same (vanishing) roughness and growth exponents, we prove that non-linearity produces non-trivial logarithmic corrections to the diffusive scaling and to the bulk diffusion coefficient. In fact, *we propose these corrections as a distinguishing feature between the EW and the AKPZ universality classes for* $2d$ *stochastic growth*. It would be extremely interesting to find (numerical and/or analytical) evidence of analogous logarithmic super-diffusivity for discrete growth models as those in [28–30], that are conjectured to have the same qualitative features as the AKPZ equation, or for the equation with non-linearity given by $[(\partial_{x_1}H)^2 - a(\partial_{x_2}H)^2], a > 0$. In fact, for $a \neq 1$ the stationary state is *not Gaussian* [22] but the RG analysis of [13] suggests that the behaviour should be the same as for $a = 1$, in particular the stationary state should be asymptotically Gaussian on large scales, as indicated by the simulations in [15]. Even though log corrections to diffusivity may look too tiny to be observed, we emphasise that the predicted $(\log t)^{2/3}$-effect in $2d$ driven diffusive models has been very recently numerically measured [31].

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